Turán's Theorem for Dowling Geometries

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joint work with Rutger Campbell & Jorn van der Pol

GT Combinatorics Seminar

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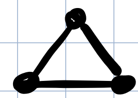
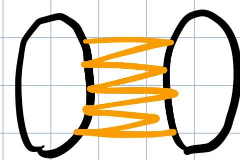
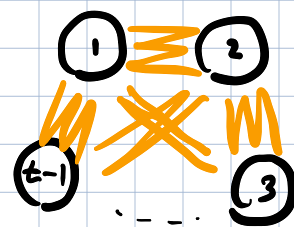
Q. # elements vs. forbidden structure in the universe.

Universe	Forbidden structure	Max # elements	
Complete graph K_n		$\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$ 	Mantel, 1910
	K_t	$\sim \binom{t-1}{2} \cdot \left(\frac{n}{t-1}\right)^2$ 	Turán, 1941

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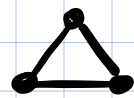
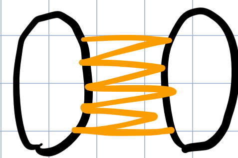
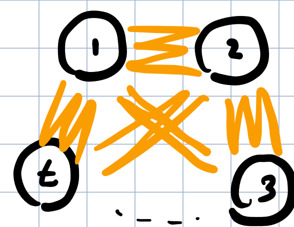
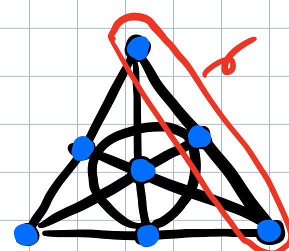
Complete graph K_n		$\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$ 	Mantel, 1910
	K_t	$\sim \binom{t-1}{2} \cdot \left(\frac{n}{t-1}\right)^2$ 	Turán, 1941
$\mathbb{P}^{n-1}(\mathbb{F}_q)$	$\mathbb{P}^{t-1}(\mathbb{F}_q)$	$\dfrac{q^n - q^{n-t+1}}{q-1}$	Bose – Burton '66

$q = 2$
$n = 3$
$t = 2$



We can re-write Turán & Bose–Burton Theorems using matroids

$M(G)$:= matroid associated with graph G

$PG(n-1, q)$:= $\mathbb{P}^{n-1}(\mathbb{F}_q)$

$ex(M, N)$:= max size of $M' \subseteq M$ s.t. M' is N-free.

Turán : $ex(M(K_n), M(K_t)) \sim \binom{t-1}{2} \cdot \left(\frac{n}{t-1}\right)^2$

Bose–Burton : $ex(PG(n-1, q), PG(t-1, q)) = \dfrac{q^n - q^{n-t+1}}{q-1}$

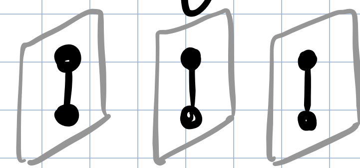
Q. $ex(M_n, N) = ?$

"Universal" rank-n matroid M_n
that contains all rank-n matroids

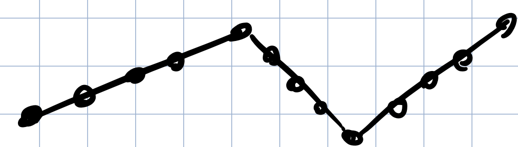
Thm (Kahn – Kung '82)

There are only 5 universal models for matroids:

- Projective geom. $PG(n-1, q)$
- Dowling geom. $Q_n(\Gamma)$
- Free matroids $U_{n,n}$
- Matchstick geom.



- Origami geom.



Dowling Geometries $Q_n(\Gamma)$: Matroids associated with multi-graphs edge-labelled by a finite group Γ

$\Gamma = \mathbb{Z}_1 \Rightarrow$ Graphs, $Q_{n-1}(\mathbb{Z}_1) = M(K_n)$

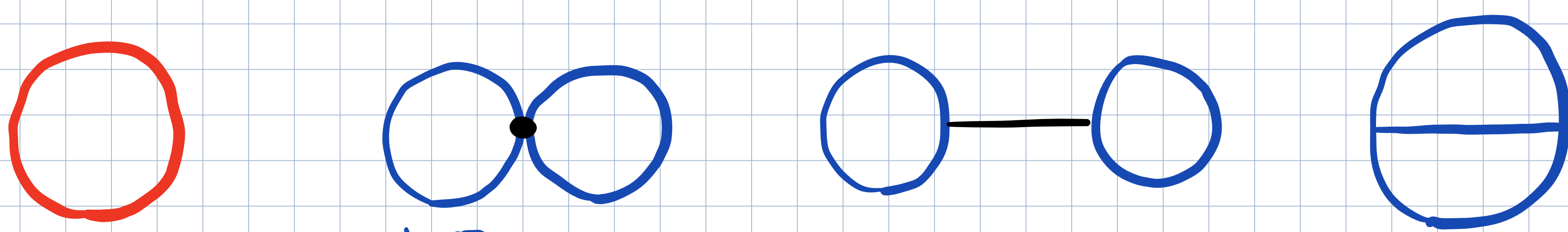
$\mathbb{Z}_2 \Rightarrow$ Signed Graphs

Group-labelled graphs & Frame matroids (Zaslavsky '8,90s)

Biased graph $(G, \mathcal{B})$: graph G & collection $\mathcal{B}$ of cycles satisfying "θ-property"

$C \in \mathcal{B}$ is called a balanced cycle

Frame matroid of $(G, \mathcal{B})$ is the matroid on E whose circuits are :



$\in \mathcal{B}$ $\notin \mathcal{B}$

eg. $\mathcal{B} =$ all cycles $\Rightarrow$ $FM(G, \mathcal{B}) =$ cycle matroid $M(G)$

$\mathcal{B} = \phi$ $\Rightarrow$ $FM(G, \mathcal{B}) =$ bicircular matroid

G : graph with a reference orientation $\vec{G}$

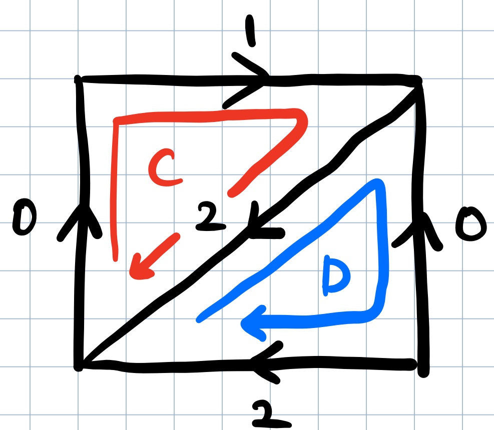
$\varphi : E \setminus \text{loops} \longrightarrow \Gamma$

Cycle $C = e_1 e_2 \cdots e_\ell$ $(\ell \geq 2)$ is balanced if

$$\varphi(C) = \varphi(e_1)^{\sigma_1} \times \varphi(e_2)^{\sigma_2} \times \cdots \times \varphi(e_\ell)^{\sigma_\ell}$$

is the identity, where $\sigma_i = \begin{cases} 1 \\ -1 \end{cases}$ whether $\vec{e_i}$ agrees C or not.

e.g. $\mathbb{Z}_3$



$$\varphi({\color{red}C}) = 0 + 1 + 2 = 0$$

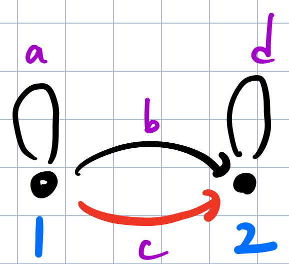
$$\varphi({\color{blue}D}) = (-2) + (-0) + 2 = 0$$

K_n^{Γ} : $V = \{1, 2, 3, \dots, n\}$

$\forall (i, j)$ with $i < j$, $\exists\, |\Gamma|$ edges from i to j labelled by the elements of Γ

$\exists$ a loop attached to each vertex i

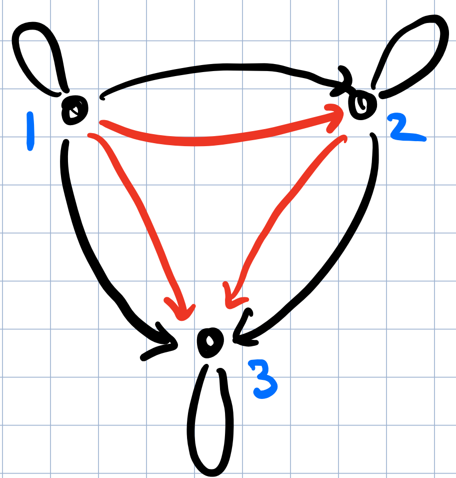
$K_2^{\mathbb{Z}_2}$



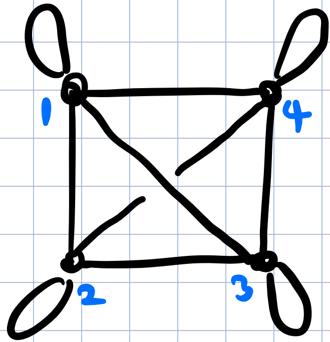
No balanced cycles

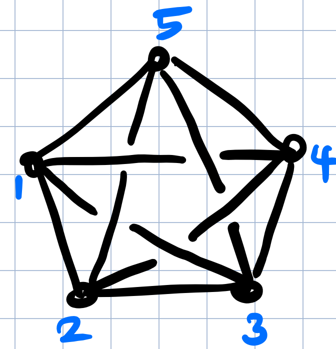
Circuits: $\binom{\{a, b, c, d\}}{3}$ $\Longrightarrow$ $FM(K_2^{\mathbb{Z}_2})$ is $U_{2,4}$

$K_3^{\mathbb{Z}_2}$



$$\boxed{Q_n(\Gamma) := FM(K_n^{\Gamma})}$$

$FM($ 

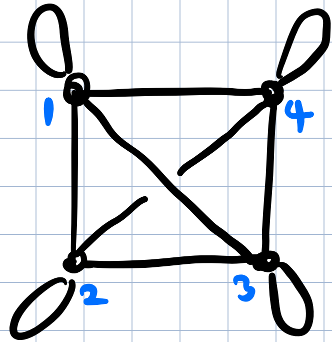
$) \simeq M($ 

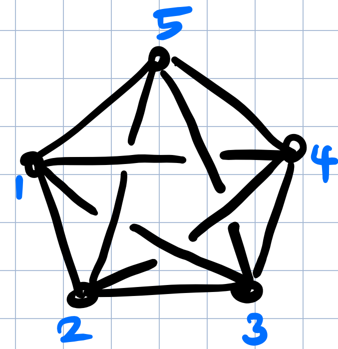
$)$

$K_4^{\mathbb{Z}_1}$

$$ex\big(Q_{n-1}(\mathbb{Z}_1),\ Q_{t-1}(\mathbb{Z}_1)\big) = ex\big(M(K_n),\ M(K_t)\big)$$

$$\approx \binom{t-1}{2} \cdot \left(\frac{n}{t-1}\right)^2$$

$\mathrm{FM}\Big($ 

$\Big) \simeq M\Big($ 

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$$\approx \binom{t-1}{2} \cdot \left(\frac{n}{t-1}\right)^2$$

Thm (Campbell, K., van der Pol) If $n \ge t \ge 3$ & $\Gamma \ge \Gamma' \neq \mathbb{Z}_1$,

$$\mathrm{ex}\big(Q_n(\Gamma),\ Q_t(\Gamma') \big) = |Q_n(\Gamma)| - n + t - 1$$

$$Q_t(\mathbb{Z}_1) = M(K_{t+1})$$

Turán density $$\pi(\Gamma, N) := \lim_{n\to\infty} \frac{ex(Q_n(\Gamma), N)}{|\Gamma| \cdot \binom{n}{2}}$$

Turán's thm. $$\pi(\mathbb{Z}_1, M(K_t)) = \frac{t-2}{t-1}$$

$$\pi(\Gamma, Q_n(\Gamma')) = 1 \qquad \text{if } \Gamma \geq \Gamma' \ncong \mathbb{Z}_1$$

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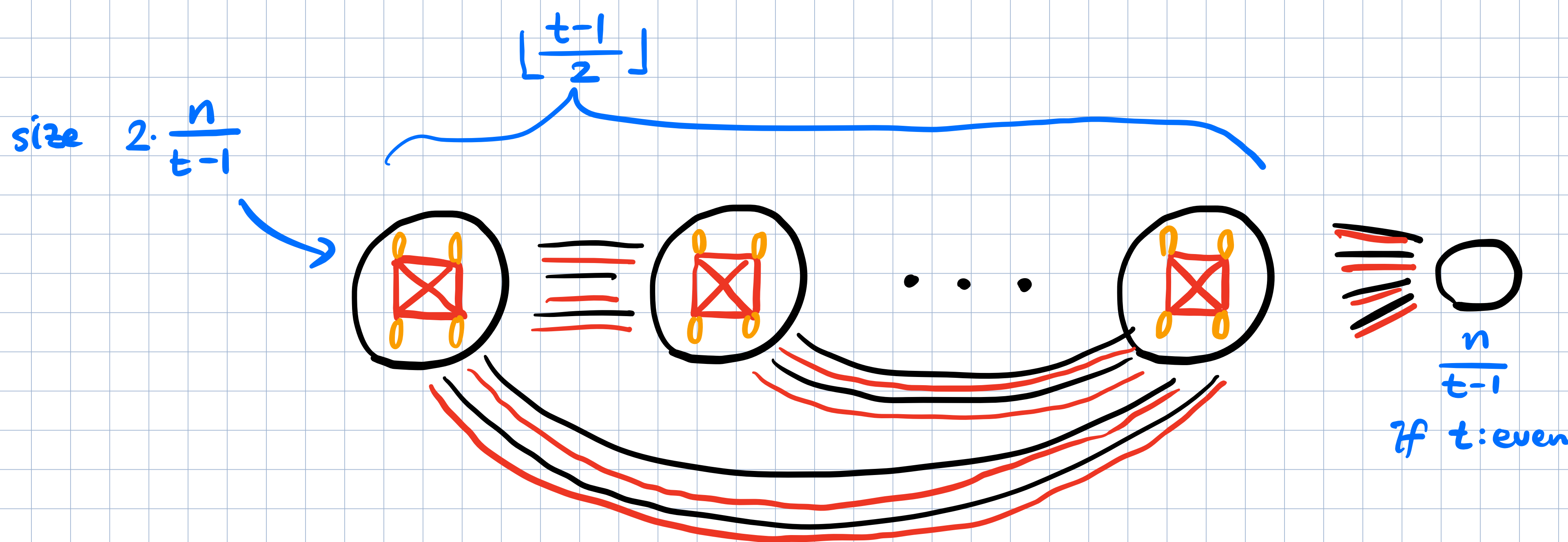
$\pi(\Gamma, Q_n(\Gamma')) = 1 \qquad$ if $\Gamma \geq \Gamma' \ncong \mathbb{Z}_1$

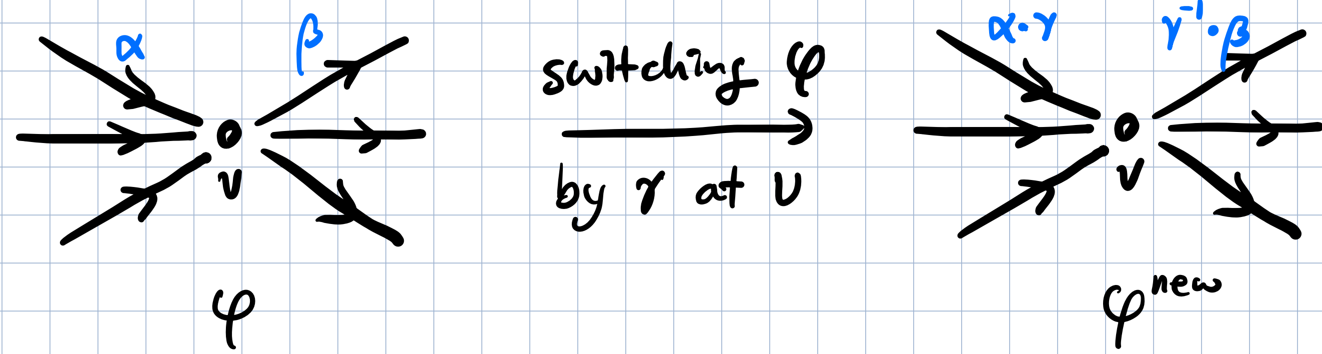
<u>Thm</u> (C−K.−vdP) If $t \geq 5$ & Γ: finite group,

$$\pi(\Gamma, M(K_t)) = \frac{t-2}{t-1}$$

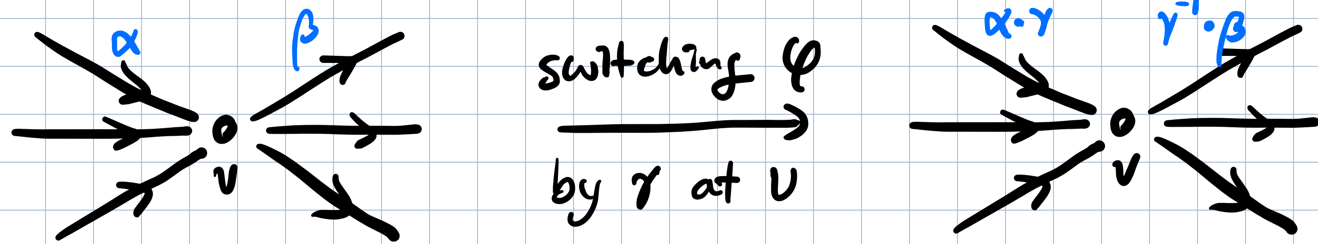
Moreover,

$$ex(Q_n(\mathbb{Z}_2), M(K_t)) = \frac{t-2}{t-1}n^2 + \left\lfloor \frac{t-2}{2} \right\rfloor \frac{1}{t-1} n + O(t^3)$$





$\therefore$ Two switching-equivalent Γ-labelled graphs induces the same frame matroid.



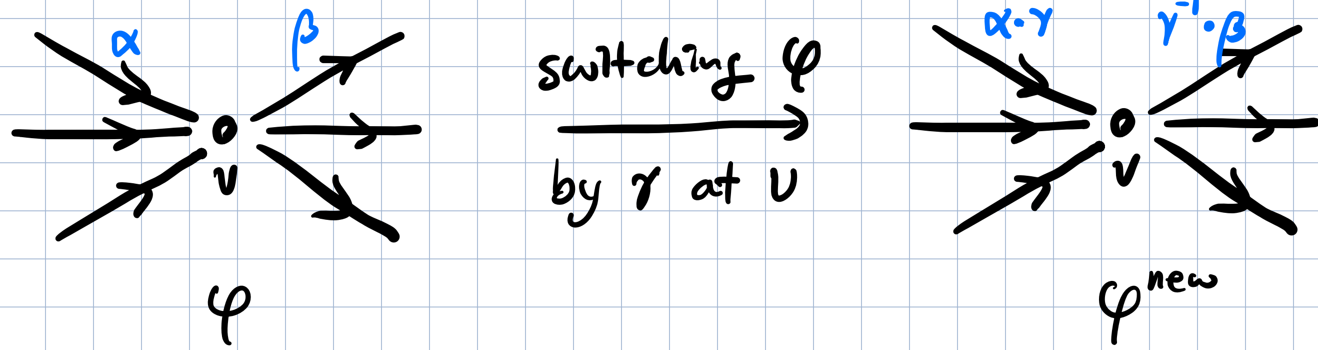
φ $\qquad\qquad$ φ^{new}

$\therefore$ Two switching-equivalent Γ-labelled graphs induces the same frame matroid.

$G, H \subset K_n^{\Gamma}$

G has a balanced H-copy if G has a subgraph switching-isomorphic to H.

<u>Lem</u> G has a balanced H-copy $\Rightarrow$ $FM(G) \supset FM(H)$



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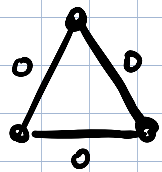
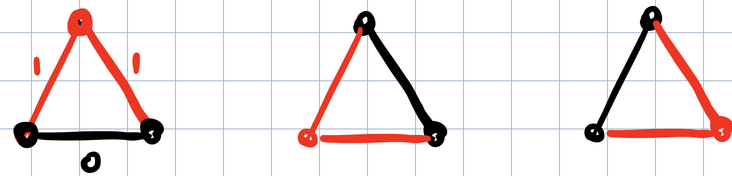
<u>Lem</u> G has a balanced H-copy $\Rightarrow$ $FM(G) \supset FM(H)$

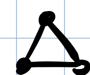
Let $H_1, H_2, \cdots, H_m$ be all Γ-labelled graphs s.t. $FM(H_i) = FM(H)$ up to switching-isomorphism.

<u>Lem</u> $FM(H) \subset FM(G)$ $\Rightarrow$ G has a balanced H_i-copy for some i.

eg. $\Gamma = \mathbb{Z}_2$

$H =$  $\xleftrightarrow{\text{switching}}$ 

$H_1 = H$, $H_2 =$ 

$\therefore$ $G < K_n^{\mathbb{Z}_2}$ has no balanced $\{$  ,  $\}$ - copy

iff $FM(G)$ has no circuit of size 3.

Thm If $t \geq 5$ & Γ: finite group,

$$\pi(\Gamma, M(K_t)) = \frac{t-2}{t-1}$$

Thm (Erdős – Stone – Simonovitz)

$$\pi_{\mathrm{graph}}(H) = \frac{\chi(H)-2}{\chi(H)-1}$$

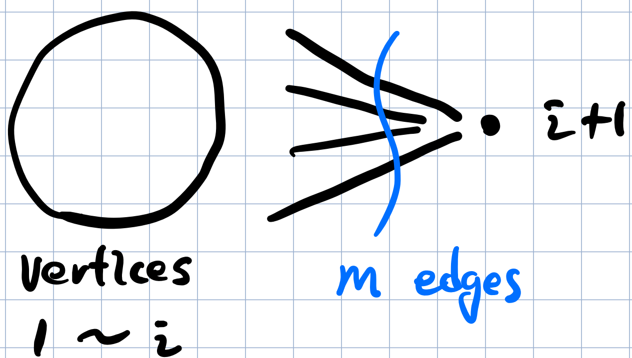
Cor

$$\pi(\Gamma, M(H)) \leq \frac{\chi(H)-2}{\chi(H)-1}$$

Prop $ex(Q_n(\Gamma), N) \le \frac{|\Gamma|}{|\Gamma'|} \cdot ex(Q_n(\Gamma'), N)$ if $\Gamma' \le \Gamma$

pf

$G \subset K_n^\Gamma$ realizing extremal example, so no N-submatroid



$\xrightarrow{\text{switchg}}$ $\ge \frac{|\Gamma'|}{|\Gamma|} \cdot m$ edges are labelled by Γ'.

$\square$

Prop $\quad ex(Q_n(\Gamma), N) \leq \dfrac{|\Gamma|}{|\Gamma'|} \cdot ex(Q_n(\Gamma'), N) \quad$ if $\Gamma' \leq \Gamma$

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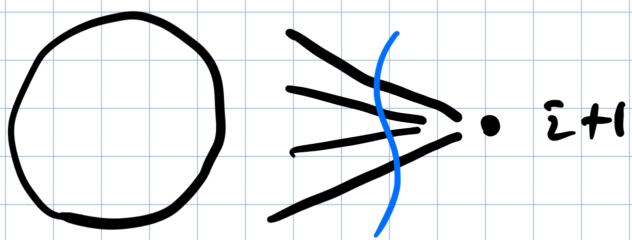


$\xrightarrow{\text{switchg}}$ $\geq \dfrac{|\Gamma'|}{|\Gamma|} \cdot m$ edges are labelled by Γ'.

$\square$

Cor

$$\pi(\Gamma, M(H)) \leq \frac{\chi(H)-2}{\chi(H)-1}$$

pf) $\quad ex(Q_n(\Gamma), M(H)) \leq |\Gamma| \cdot ex(Q_n(\mathbb{Z}_1), M(H))$

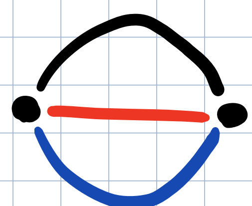
$\qquad\qquad \leq |\Gamma| \cdot ex_{graph}(n+1, H) \qquad \square$

Cor 2 $\quad \pi(\Gamma, N) = 0 \iff N = M(\text{bipartite graph})$

ESS-type bound is not tight.

Thm (Mantel's thm for Dowling geometries)

$$\mathrm{ex}(Q_n(\Gamma), M(K_3)) = \begin{cases} \left\lfloor \frac{(n+1)^2}{4} \right\rfloor & \text{if } \Gamma = \mathbb{Z}_1 \\ \left\lfloor \frac{n^2}{2} \right\rfloor & \mathbb{Z}_2 \\ \left\lceil \frac{n^2}{2} \right\rceil & \mathbb{Z}_3 \\ 2\binom{n}{2} & |\Gamma| \geq 4 \end{cases} \quad \Longrightarrow \quad \begin{array}{c} \pi(\Gamma, M(K_3)) \\ 1/2 \\ 1/2 \\ 1/3 \\ 2/|\Gamma| \end{array}$$

FM() $\simeq M(K_3)$

Thm If $\Gamma = \mathbb{Z}_2$ or Γ has no element of order two,

$$\pi(\Gamma, M(K_4)) = \frac{2}{3}$$

Otherwise, $$\pi(\Gamma, M(K_4)) \geq \frac{|\Gamma|}{2|\Gamma|-1}$$

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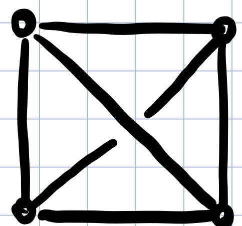
Otherwise,

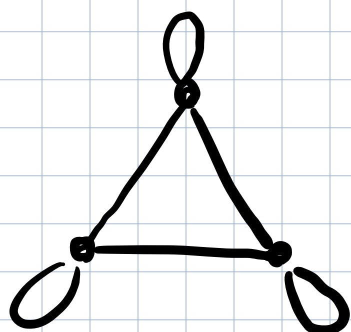
$$\pi(\Gamma, M(K_4)) \geq \frac{|\Gamma|}{2|\Gamma|-1}$$

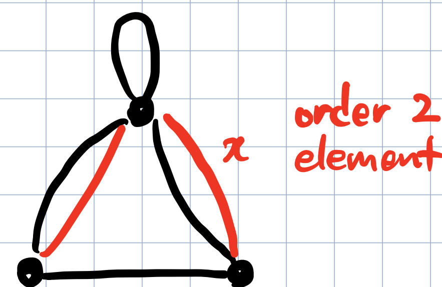
$$\frac{4}{7} \leq \pi(\mathbb{Z}_2 \times \mathbb{Z}_2, M(K_4)) \leq \frac{7}{12} \approx 0.5833$$

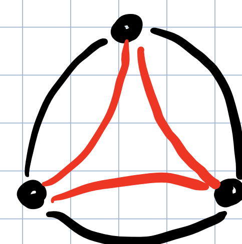
$$\wedge$$

$$0.6154 \approx \frac{8}{13} \leq \pi(\mathbb{Z}_4, M(K_4)) \leq \frac{2}{3}$$









When $\pi(\Gamma, H) = \dfrac{\chi(H)-2}{\chi(H)-1}$?

A pair (Γ, H) is critical if there is no $H' \subset K_m^\Gamma$ s.t.

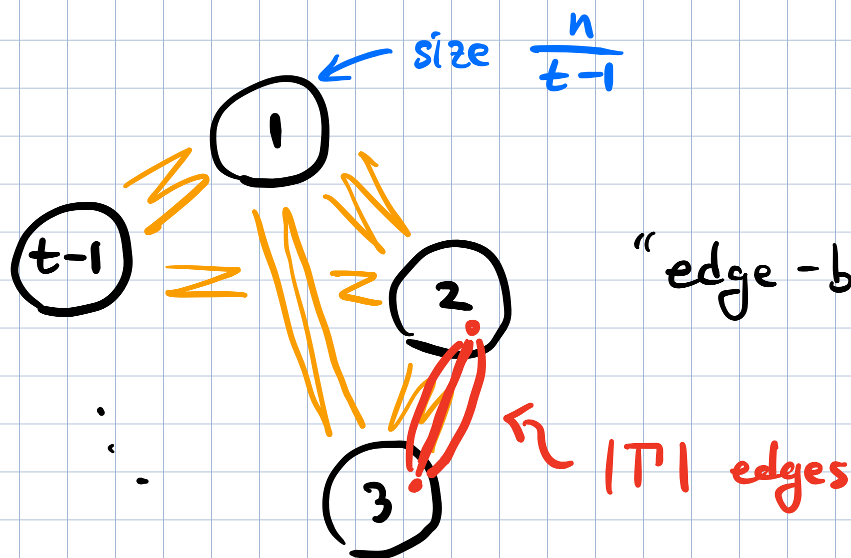
① $FM(H) \simeq FM(H')$

② H' is loopless

③ $\chi(\underline{H'}) < \chi(H)$

Prop If (Γ, H) is critical, $\pi(\Gamma, M(H)) = \dfrac{\chi(H)-2}{\chi(H)-1}$

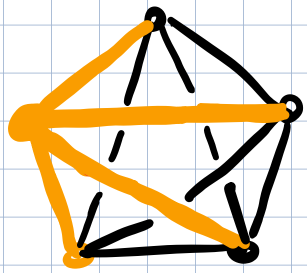
pf) $t := \chi(H)$



"edge-blow-up" of $T_{n,t-1}$

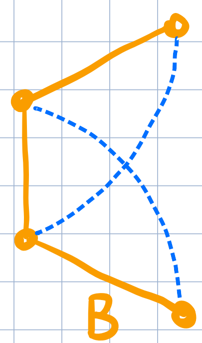
eg. (Γ, K_t) with $t \geq 5$ is critical $\therefore \pi(\Gamma, K_t) = \frac{t-2}{t-1}$

$M(K_t)$ has a basis B s.t. the circuit in $B+e$ has size 3 for any $e \in E \setminus B$.

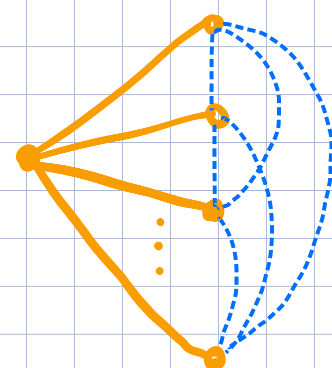


loopless $H' \subset K_t^{\Gamma}$ s.t. $FM(H') = M(K_t)$

$\Rightarrow$ B should be a tree.



$\Rightarrow$ B is a t-star.

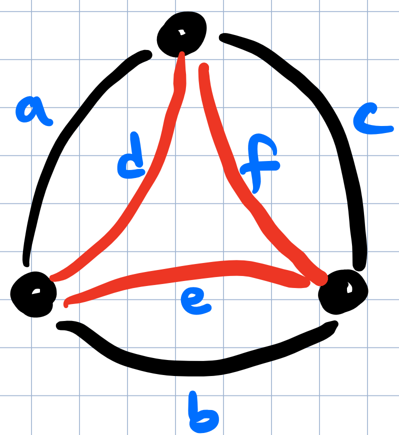


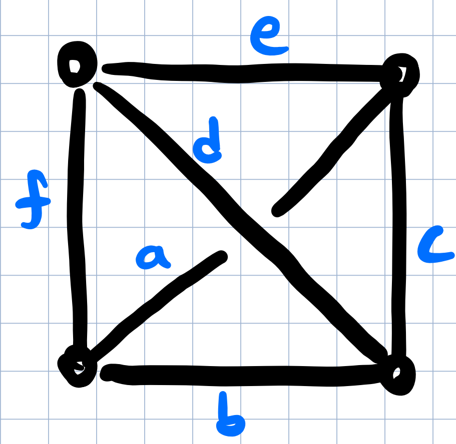
$\Rightarrow$ H' is complete.

$\therefore$ K_t is Γ-critical.

eg2. K_4 is not $\mathbb{Z}_2$-critical.

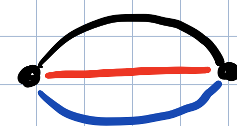
$$\pi(\mathbb{Z}_2, K_4) = \frac{2}{3}$$

FM(

) $\simeq$ M(

)

For $|\Gamma| = 4$, K_3 is not Γ-critical



$$\pi(\Gamma, K_3) = \frac{1}{2}$$

Questions.

1. Determine $\pi(\Gamma, M(K_4))$ for $\Gamma \ncong \mathbb{Z}_2$ that contains an element of order 2.

$$\boxed{\begin{array}{ccc} & \dfrac{4}{7} \leq \pi(\mathbb{Z}_2 \times \mathbb{Z}_2,\ M(K_4)) \leq \dfrac{7}{12} & \approx 0.5833 \\ & \wedge & \\ 0.6154 \approx & \dfrac{8}{13} \leq \pi(\mathbb{Z}_4,\ M(K_4)) \leq \dfrac{2}{3} & \end{array}}$$

2. $|\Gamma_1| = |\Gamma_2|$ & $|\mathrm{Aut}(\Gamma_1)| < |\mathrm{Aut}(\Gamma_2)| \Rightarrow \pi(\Gamma_1, N) \leq \pi(\Gamma_2, N)$?

3. For H: bipartite, can you compute

$$ex(Q_n(\Gamma), M(H)) = \Theta(n^{*}) \ ?$$

Questions.

1. Determine $\pi(\Gamma, M(K_4))$ for $\Gamma \not\cong \mathbb{Z}_2$ that contains an element of order 2.

$$\boxed{\begin{array}{c} \frac{4}{7} \leq \pi(\mathbb{Z}_2 \times \mathbb{Z}_2,\ M(K_4)) \leq \frac{7}{12} \approx 0.5833 \\ \wedge \\ 0.6154 \approx \frac{8}{13} \leq \pi(\mathbb{Z}_4,\ M(K_4)) \leq \frac{2}{3} \end{array}}$$

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3. For H: bipartite, can you compute

$$ex(Q_n(\Gamma), M(H)) = \Theta(n^{*}) \ ?$$

Thank you